Delay Curves for Calls Served at Random

By JOHN RIORDAN

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This paper presents curves and tables for the probability of delay of calls served by a simple trunk group with assignment of delayed calls to the trunks at random and with pure chance call input. These are contrasted with the classic results of Erlang ("Erlang C") which are based on service in order of arrival. Trunk holding times for both have an exponentiol distribution. The theoretical development for computation of the curves is directed to the determination of the moments, which seem to be a natural means of simplification.

1. INTRODUCTION

One of the classic results in the study of telephone traffic is the formula for delay given by the Danish engineer A. K. Erlang[1] in 1917. This is for random call input to a fully accessible simple trunk group with the trunk holding time exponential and calls served in the order of arrival. A proof for this formula and a set of curves for its use have been given by E. C. Molina.[2]

In many switching systems it is not feasible to fully realize this ethical ideal of first come, first served, and it has long been of interest to determine delays on another basis. The contrasting assumption is of calls picked at random, which is again an idealization but in large offices appears to be called for, as a bound for the service actually given.

The first attempt to formulate the last seems to be that of J. W. Mellor.[3] While his basic formulation is incomplete, it offers a useful approximation to the complete results, particularly in the most interesting region of heavy traffic, and will be referred to here as the "Mellor approximation." A complete formulation due to E. Vaulot[4] appeared in 1946 and included both the fundamental differential recurrence relation and formulas for delay probabilities for small delays. For completeness, these are repeated below. F. Pollaczek[5] has given a development of Vaulot's work directed toward determining an asymptotic delay formula.

Considerable further theoretical work has been necessary to obtain the results given here. Vaulot's differential recurrence relation, which formulates the probability of delay at least t of a call which arrives when n other calls are waiting, has no simple solution. By approximate methods, it was possible to use a differential analyser to determine these probabilities for small values of n. But it was not feasible in this way to cover the whole range of interest, and these results were supplemented by approximations for large n, which are described below. Finally the delay for an arbitrary call was obtained by summing on n.*

These results are not reported here, because the attempt to verify the accuracy attained led to formulation of the moments of the delay curves and this in turn to the representation of the curves as sums of exponential curves, with great simplification of the calculations required. As will appear, two exponentials furnish a sufficient approximation except for heavy traffic.

2. DELAY CURVES

The delay distribution on calls delayed for occupancy levels (defined below) from 0.1 to 0.9 in steps of 0.1 is shown in Fig. 1. The abscissae are derived time units which seem to be natural to the problem: $u = ct/h$, with c the number of trunks and h the average holding time. The ordinates, on a logarithmic scale, are conditional probabilities that a call delayed will be delayed at least u, that is, values of a function $F(u)$; the logarithmic scale is chosen to emphasize the dominantly exponential character of the curves. The occupancy level α is the ratio a/c where a is the average call input in average holding time h.

Fig. 1 is a master curve for all eventualities and may be changed to working curves for various sizes of trunk groups. For the construction of these curves Table I, from which Fig. 1 was made, and which also compares present results with those for calls served in order of arrival, is convenient. A more elaborate table will be given later. For the convenience of the reader, it may be noticed that for order of arrival service $F(u) = e^{-u(1-\alpha)}$.

The striking feature of Table I is the increase in delay time for random service, which becomes more pronounced with decreasing $F(u)$ and increasing occupancy (or traffic) level, α. The increase throughout the table is an effect of the limitation to small values of $F(u)$. For given

* Thanks are due to George W. Abrams for directing this work, to Dr. Richard W. Hamming for transforming the equations into forms suitable for the differential analyser and for supervising its operation, and to Miss Catherine Lennon for a great deal of calculation.

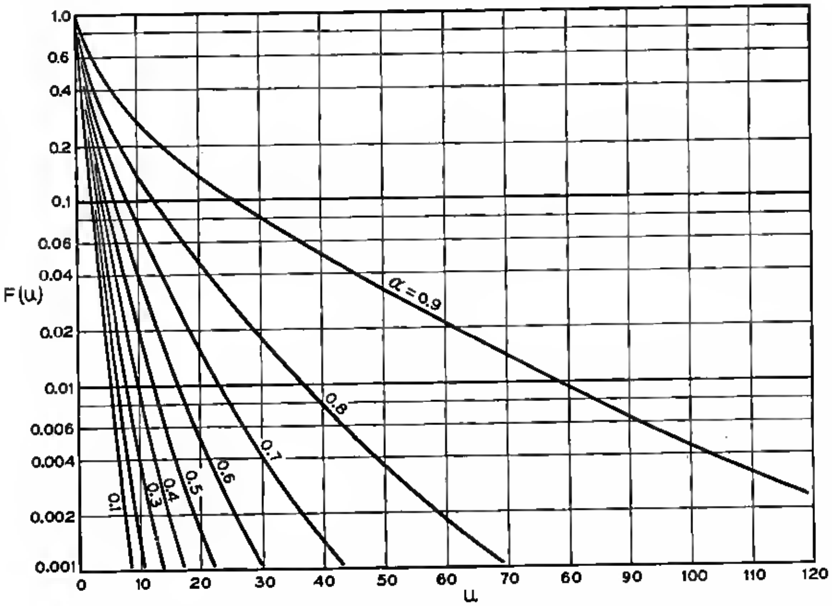

Fig. 1—Delay curves for random service. $F(u)$ = conditional probability of delay at least u; $u = ct/h$, c = no. trunks, h = av. holding time, a = call input in time h, $\alpha = a/c$.

α, the delay curves for order of arrival and random service include the same area, which is in fact equal to the mean delay (of calls delayed), $(1 - \alpha)^{-1}$. Since $F(0) = 1$ for both, and the random service curve decreases more slowly for large u, the curves must intersect at some point, say for $u = u_0$; for $u < u_0$, the o.a. curve must be above the ran-

TABLE I — DELAY-TIME AND RANDOM SERVICE

Delay Times, u, for given $F(u)$ and α and for order of arrival (o.a.) and random service.

α	$F = 0.1$		$F = 0.01$		$F = 0.001$	
	o.a.	Random	o.a.	Random	o.a.	Random
0.1	2.56	2.58	5.12	5.47	7.68	8.60
0.2	2.88	2.91	5.76	6.57	8.63	10.68
0.3	3.29	3.34	6.58	8.05	9.87	13.35
0.4	3.84	3.91	7.68	10.04	11.52	16.95
0.5	4.61	4.68	9.21	12.89	13.82	22.09
0.6	5.76	5.82	11.51	17.25	17.27	29.97
0.7	7.68	8.28	15.36	23.14	23.03	43.33
0.8	11.51	12.57	23.03	36.80	34.54	70.29
0.9	23.03	25.99	46.05	77.24	69.08	156.63

dom curve. This is shown in Fig. 2 for $\alpha = 0.9$, but the logarithmic scale for $F(u)$ obscures the equality of area.

The character of the comparison may be clearer if the picture is changed. Consider a department store counter with c clerks (corresponding to c trunks) in attendance. The time for a sale corresponds to the trunk holding time, and the rate of arrival of customers is like tbat of call input. For service in order of arrival customers are given serially numbered tickets on arrival; for random service, these tickets may be supposed drawn from a hat, or numbered from a series of random numbers, or since aggressiveness and the clerks' attention are subject to devious rule, it may be that no attention at all to order of service is equivalent to random service.

The fact that tbe average delay is independent of the order of service may be explained roughly by saying that the average rate at wbich waiting lines are removed depends only on the average rate of arrival of customers and the rate at which they are served. Notice however that service at random causes more variable delays (the second and all higher moments are larger than for order of arrival service). Thus with random service tbe proportion of waiting customers receiving quick

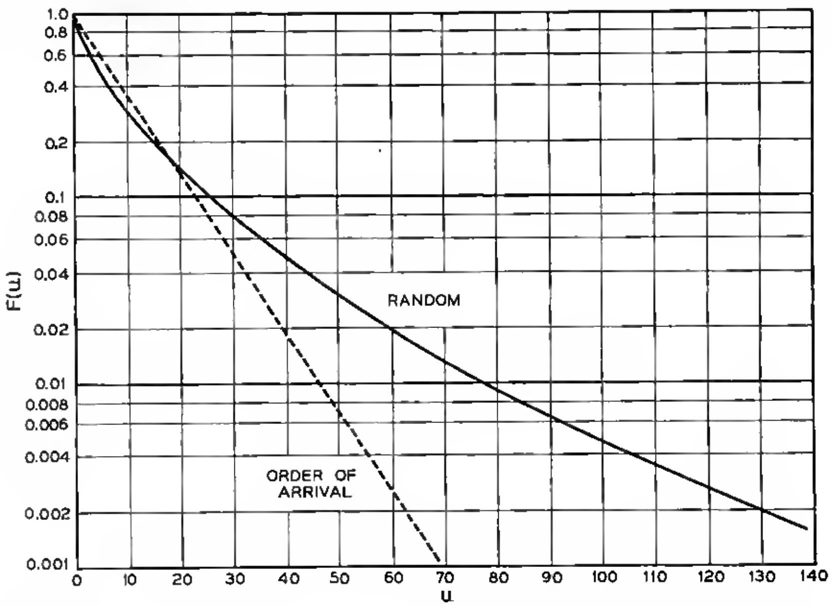

Fig. 2—Comparison of delay curves for order of arrival and random service; $\alpha = 0.9$.

service is increased (over order of arrival) but this is achieved at the cost of making other customers wait much longer.

Service in order of arrival has the advantage to the customer that his delay is independent of all who come after him, and this is particularly appreciated in times of heavy crowding when long delays are possible for random service. In Table I, these crowded conditions correspond to small values of $F(u)$ or large values of α, or both. In this picture it seems intuitively clear that much longer delays are possible for random service, for those unlucky customers who keep missing their turn. (Of course, a more realistic model would also include the effects of customers leaving before service, a factor of considerable telephone interest also.)

As noted at the start of this section, $F(u)$ is a conditional probability, the probability of delay at least u of a call that is surely delayed. To obtain unconditional probabilities of delay, $F(u)$ is multiplied by the probability that all trunks are busy, which is the probability that a call is delayed. This probability is given by a well-known formula due to Erlang and customarily written as

$$C(c, a) = \frac{a^c}{(c-1)!\,(c-a)}\left[1 + \frac{a}{1!} + \frac{a^2}{2!} + \cdots + \frac{a^{c-1}}{(c-1)!} + \frac{a^c}{(c-1)!\,(c-a)}\right]^{-1}$$

Tables of this function are available*.

Finally it may be noticed here that for random service and light traffic (roughly, α less than 0.7), with sufficient approximation

$$F(u) = \tfrac{1}{2}(y_1 e^{-u(1-\alpha)y_1} + y_2 e^{-u(1-\alpha)y_2})$$

with $y_1 = 1 - \sqrt{\alpha/2}$, $y_2 = 1 + \sqrt{\alpha/2}$.

* But there seems to be no extensive tabulation. However, the table for the Erlang B function made by Conny Palm (Stockholm, 1947) may be used with the relations

$$\frac{1}{C(c, a)} = \frac{1}{B(c, a)} - \frac{1}{B(c-1, a)}$$

$$= \frac{a}{c} + \frac{1 - (a/c)}{B(c, a)}$$

Notice that $C(c, a)$ also has the recurrence relation

$$\frac{1}{C(c, a)} = \frac{-1}{c - 1 - a} + \frac{(c-a)(c-1)}{a(c-1-a)C(c-1, a)}$$

3. BASIC FORMULATION

As noted above, the following notation is used: c is the number of trunks, h is the average holding time (the distribution of holding times is exponential) and a is the average number of calls arriving in time interval h. Then, if $F_n(t)$ is the probability of delay at least t of a call arriving when n other calls are waiting, the differential recurrence relation given by Vaulot is

$$\frac{dF_n(t)}{dt} = \frac{n}{n+1}\frac{c}{h}F_{n-1}(t) - \frac{c+a}{h}F_n(t) + \frac{a}{h}F_{n+1}(t) \tag{1}$$

This may be derived as follows. Consider the interval dt after the epoch of arrival of the call in question. In this interval three events may occur: (i) a call may arrive, (ii) a trunk may be released, or (iii) neither of these. The probability of a call arrival is $(a/h)dt$ and if a call arrives the delay function is $F_{n+1}(t - dt)$. The probability of a trunk release, because of the assumption of exponential holding time, is $(c/h)dt$, and if a trunk is released the number of waiting calls is reduced by one; the probability that the call seizing the waiting trunk will not be the call in question is $n/(n + 1)$. Finally the probability of the third event is $1 - (c + a)dt/h$. All this is summarized in the differential relation

$$F_n(t) = \frac{a}{h}\,dt\,F_{n+1}(t - dt) + \frac{n}{n+1}\frac{c}{h}\,dt\,F_{n-1}(t - dt) + \left(1 - \frac{c+a}{h}\,dt\right)F_n(t - dt)$$

Passing to the limit gives equation (1).

Using new variables: $u = ct/h$, $\alpha = a/c$, equation (1) may be written more simply as

$$\frac{dF_n(u)}{du} = \frac{n}{n+1}F_{n-1}(u) - (1+\alpha)F_n(u) + \alpha F_{n+1}(u) \tag{1a}$$

This equation is a mixed differential-difference equation of the first order as a differential equation and of the second order as a difference equation; hence three boundary relations are required. For the differential part, it is clear that $F_n(0)$, which is the probability of some delay of the test call, is unity for all n in question, that is, for all integral non-negative n. Also $F_n(u) \equiv 0$ for all negative n, is an obvious necessity, and, since F_n is a distribution function $F_n(\infty) = 1$. Finally the third

condition may be stated as

$$\lim_{n=\infty} F_n(u) = 1, \qquad \text{all } u$$

The probability of delay at least u of an arbitrary call is the sum on n of the product of the probability that n calls are waiting when the call arrives and the probability, $F_n(u)$, that for this condition the call is delayed at least u. The first probability (for statistical equilibrium) is known to be

$$(1 - \alpha)\, C(c, a)\alpha^n$$

where $C(c, a)$, as stated above, is the probability tbat all trunks are busy; $(1 - \alpha)C(c, a)$ is the probability that all trunks are busy and no calls are waiting. Hence the probability in question, say $f(u)$, is given by

$$f(u) = (1 - \alpha)C(c, a) \sum_0^\infty \alpha^n F_n(u)$$

or by

$$f(u) = C(c, a)F(u)$$

if

$$F(u) = (1 - \alpha) \sum_0^\infty \alpha^n F_n(u) \tag{2}$$

$F(u)$, like $F_n(u)$, is then a conditional probability, the probability at least u of a delayed call. Notice that, consistent with this, $F(0) = 1$.

It is interesting to notice that Mellor's basic equation, which in present notation may be written as

$$\frac{dG_n(u)}{du} = -\frac{1}{n + 1}\, G_n(u), \tag{3}$$

follows from (1) if first it is supposed that $F_{n-1}(u) = F_n(u) = F_{n+1}(u)$ and then, for clarity, G_n replaces F_n. Hence, as indicated by the third boundary condition, it may be expected to be useful for large values of n. Its solution is

$$G_n(u) = e^{-u/(n+1)} \tag{4}$$

A somewhat better approximation may be determined by the MacLaurin series obtained by repeated differentiation of (1a) and evaluation

at $u = 0$; this is as follows

$$F_n(u) \approx 1 - \frac{u}{n+1} + \frac{\alpha}{2}\frac{(u)^2}{(n+1)} - \frac{\alpha(2\alpha-1)}{3!}\frac{(u)^3}{(n+1)} + \frac{\alpha(2\alpha-1)(3\alpha-2)}{4!}\frac{(u)^4}{(n+1)} - \cdots \tag{5}$$

But this is the same* as:

$$F_n(u) \approx [1 - (1-\alpha)u/(n+1)]^{1/(1-\alpha)} \tag{5a}$$

As α approaches unity, (5a) approaches (4). Equation (5a) has been used, for large values of α, in the direct computations mentioned above.

It may also be noted that for $\alpha = 0$, equation (1a) has the solution (now writing $F_n(u, \alpha)$ for $F_n(u)$)

$$F_n(u, 0) = \phi(u, n) - \frac{u}{n+1}\phi(u, n-1) \tag{6}$$

where $\phi(u, n)$ is the Poisson sum

$$e^{-u}\left(1 + u + \frac{u^2}{2!} + \cdots + \frac{u^n}{n!}\right)$$

Finally, for completeness, note that for small values of u, the MacLaurin series for $F(u)$ is

$$\begin{aligned} F(u) = 1 &- u\,\frac{1-\alpha}{\alpha}\log\frac{1}{1-\alpha} \\ &+ \frac{u^2}{2}(1-\alpha)\left[2 - \frac{1-\alpha}{\alpha}\log\frac{1}{1-\alpha}\right] \\ &- \frac{u^3}{6}(1-\alpha)\left[1 + 3\alpha - (1-\alpha)\log\frac{1}{1-\alpha} - \sum_1^\infty \frac{\alpha^n}{n^2}\right] \end{aligned} \tag{7}$$

4. MOMENTS

The k'th moment (about the origin) of the delay density function which is $-F'(u)$ ($F(u)$ itself is a distribution function is defined as

$$\begin{aligned} M_k &= \int_0 u^k[-F'(u)]\,du, \\ &= k\int_0 u^{k-1}F(u)\,du, \qquad k > 0, \end{aligned} \tag{8}$$

the last by integration by parts.

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* G. W. Abrams is due credit for noticing this.

Following (2), this may also be written as

$$M_k = (1 - \alpha) \sum_0^\infty \alpha^n m_{n,k}, \tag{9}$$

with

$$\begin{aligned} m_{n,k} &= \int_0^\infty u^k[-F'_n(u)]\,du, \\ &= k \int_0^\infty u^{k-1} F_n(u)\,du, \qquad k > 0. \end{aligned} \tag{10}$$

First, notice that

$$m_{n,0} = -\int_0^\infty F'_n(u)\,du = F_n(0) = 1;$$

hence

$$M_0 = (1 - \alpha) \sum_0^\infty \alpha^n = 1,$$

showing that $F(u)$ is properly normalized.

Next, by integrating both sides of (1a) with respect to u from 0 to ∞, and using the second form of (10) (with $k = 1$)

$$-(n + 1) = nm_{n-1,1} - (n + 1)(1 + \alpha)m_{n,1} + (n + 1)\alpha m_{n+1,1} \tag{11}$$

In the same way, after first multiplying (1a) throughout by u^{k-1}, it is found that

$$\begin{aligned} -k(n + 1)m_{n,k-1} \\ = nm_{n-1,k} - (n + 1)(1 + \alpha)m_{n,k} + (n + 1)\alpha m_{n+1,k} \end{aligned} \tag{12}$$

Unfortunately, neither (11) nor any other instances of (12) have simple solutions; nevertheless they may be used to determine M_k.

Consider first the simplest case, M_1. If (11) is multiplied throughout by α^n and summed on n, the result may be written

$$\begin{aligned} -L_{10} &= \alpha L_{11} - (1 + \alpha)L_{11} + L_{11} - L_{01} \\ &= -L_{01} \end{aligned} \tag{13}$$

where for convenience in writing and of later notation

$$L_{01} = \sum \alpha^n m_{n,1} = (1 - \alpha)^{-1} M_1$$

$$L_{11} = \sum (n + 1)\alpha^n m_{n,1}$$

$$L_{10} = \sum (n + 1)\alpha^n = D \sum \alpha^{n+1} = (1 - \alpha)^{-2}$$

and $D = d/d\alpha$. Hence

$$M_1 = (1 - \alpha)^{-1}$$

This is the mean delay of calls delayed and as mentioned above is the same as for service in order of arrival.

In the general case*, the following notation is convenient

$$L_{0k} = \sum \alpha^n m_{n,k} = (1 - \alpha)^{-1} M_k$$
$$L_{jk} = \sum (n + 1)(n + 2) \cdots (n + j)\alpha^n m_{n,k}$$

Using the relations

$$\begin{aligned} n(n + 2) \cdots (n + j) &= n(n + 1) \cdots (n + j - 1) + (j - 1)n(n + 1) \cdots (n + j - 2) \\ &+ \cdots + (j - 1)_i n(n + 1) \cdots (n + j - i - 1) + \cdots \\ &+ (j - 1)!n, \end{aligned}$$

$$\begin{aligned} n(n + 1) \cdots (n + j - 1) &= (n + 1)(n + 2) \cdots (n + j) - j(n + 1)(n + 2)(n + j - 1) \end{aligned}$$

with

$$(j - 1)_i = (j - 1)(j - 2) \cdots (j - i),$$

the summing of (12) is found to result in

$$\begin{aligned} kL_{j,k-1} &= [j - (j - 1)\alpha]L_{j-1,k} - \alpha[(j - 1)_2 L_{j-2,k} \\ &+ (j - 1)_3 L_{j-3,k} + \cdots + (j - 1)_i L_{j-i,k} + \cdots + (j - 1)!L_{1,k}] \end{aligned} \tag{14}$$

But this may be simplified by multiplying through by j and subtracting from the same equation with j replaced by $j + 1$; the result is

$$(j + 1 - j\alpha)L_{jk} - j^2 L_{j-1,k} = kL_{j+1,k-1} - jkL_{j,k-1} \tag{15}$$

Notice that for $j = 0$, $k = 1$, $L_{01} = L_{10}$, as in (13). Notice also that

$$\begin{aligned} \alpha^{j-1} L_{j0} &= \sum (n + 1) \cdots (n + j)\alpha^{n+j-1} \\ &= D \sum (n + 1) \cdots (n + j - 1)\alpha^{n+j} \\ &= D(\alpha^j L_{j-1,0}) \end{aligned}$$

so that

$$L_{j0} = jL_{j-1,0} + \alpha D L_{j-1,0} = j!(1 - \alpha)^{-j-1}$$

* This procedure is the development of a suggestion made by S. O. Rice.

Then the ratio

$$\begin{aligned} L_{0k}(L_{k0})^{-1} &= (1 - \alpha)^{k+1} L_{0k}/k! \\ &= (1 - \alpha)^k M_k/k! = R_k \end{aligned} \tag{16}$$

is the ratio of these moments to those for order of arrival service; the last relation is a definition. In the same way the ratio

$$L_{jk}(L_{j+k,0})^{-1}$$

might be considered, but to avoid fractions the following somewhat odd change of variables seems convenient:

$$\binom{j+k}{k} g_k L_{jk} = p_{jk} L_{j+k,0} \tag{17}$$

where $g_0 = g_1 = 1$ and

$$\begin{aligned} g_{2k} &= (2 - \alpha)^{2k-1}(3 - 2\alpha)^{2k-3} \cdots (k + 1 - k\alpha) \\ g_{2k+1} &= (2 - \alpha)^{2k}(3 - 2\alpha)^{2k-2} \cdots (k + 1 - k\alpha)^2 \end{aligned}$$

Notice that

$$g_{2k+1}(g_{2k})^{-1} = g_{2k}(g_{2k-1})^{-1} = (2 - \alpha)(3 - 2\alpha) \cdots (k + 1 - k\alpha) = D_{k+1}$$

the last being a definition, again.

Since

$$(1 - \alpha)L_{k,0} = kL_{k-1,0}$$

it follows from (15) that

$$\begin{aligned} (j + 1 - j\alpha)p_{jk} &- j(1 - \alpha)p_{j-1,k} \\ &= (g_k/g_{k-1})[(j + 1)p_{j+1,k-1} - j(1 - \alpha)p_{j,k-1}] \end{aligned} \tag{18}$$

By taking differences of this equation and writing

$$\begin{aligned} q_{0k} &= p_{0k} \\ q_{1k} &= p_{1k} - p_{0k} = \Delta p_{0k} \\ q_{2k} &= p_{2k} - 2p_{1k} + p_{0k} = \Delta^2 p_{0k} \\ q_{jk} &= \Delta q_{j-1,k} = \Delta^j p_{0k} \end{aligned}$$

a somewhat simpler recurrence relation is found to be as follows

$$\begin{aligned} (j + 1 - j\alpha)q_{jk} &= (g_k/g_{k-1}) \\ &[j\alpha q_{j-1,k-1} + (j + 1 + j\alpha)q_{j,k-1} + (j + 1)q_{j+1,k-1}] \end{aligned} \tag{19}$$

Since $p_{j0} = 1$, all j, $q_{00} = 1$, and $q_{j0} = 0, j \neq 0$. From these boundary conditions, it follows at once from (19) that

$$q_{jk} = 0, \qquad j > k.$$

By comparison of (17) and (16)

$$g_k R_k = p_{0k} = q_{0k}$$

A short table of the q's is as follows:

j/k	0	1	2	3
0	1	1	2	$2(2 + \alpha)$
1	0	$\alpha(2 - \alpha)^{-1}$	$4\alpha(2 - \alpha)^{-1}$	$2\alpha(18 - 5\alpha - 4\alpha^2)D_3^{-1}$
2	0	0	$2\alpha^2(3 - 2\alpha)^{-1}$	$2\alpha^2(18 - 7\alpha - 2\alpha^2)(3 - 2\alpha)^{-2}$
3	0	0	0	$6\alpha^3 D_2^2 D_4^{-1}$

Continuation of this leads to the values of R_k listed in Table II. Notice that for $\alpha = 1$, by (18)

$$\begin{aligned} p_{jk}(1) &= (j + 1)p_{j+1,k-1}(1) \\ &= (j + 1)(j + 2)p_{j+2,k-2}(1) \\ &= (j + 1)(j + 2) \cdots (j + k) \end{aligned}$$

since $g_k = 1$ for $\alpha = 1$ and $p_{j0} = 1$, all j. From this

$$p_{0k}(1) = g_k(1)R_k(1) = R_k(1) = k!$$

On the other hand, for $\alpha = 0$, $q_{jk} = 0, j > 0$ and, by (19)

$$q_{0k}(0)/g_k(0) = q_{0,k-1}(0)/g_{k-1}(0)$$

so that

$$R_k(0) = R_{k-1}(0) = R_1(0) = 1$$

5. MELLOR APPROXIMATION

It is useful to have the moments of the distribution corresponding to the Mellor approximation, since they serve as a guide. Here, following equation (4)

$$F(u) = (1 - \alpha) \sum_0^\infty \alpha^n e^{-u(n+1)^{-1}} \tag{20}$$

and

$$\begin{aligned}\exp - xM &= \sum_0^\infty M_k(-x)^k/k! \\ &= (1-\alpha)\int_0^\infty du\, e^{-xu} \sum_0^\infty (n+1)^{-1}\alpha^n e^{-u(n+1)^{-1}} \qquad (21) \\ &= (1-\alpha)\sum_0^\infty \alpha^n[1 + x(n+1)]^{-1}\end{aligned}$$

Hence

$$M_k = k!(1-\alpha)\sum_0^\infty (n+1)^k \alpha^n \qquad (22)$$

These moments are expressible in terms of polynomials associated with the distribution of permutations into classes according to the number of readings left to right necessary to find the elements in standard order.[6] Indeed the ratio

$$r_k(\alpha) = M_k\,(1-\alpha)^k/k!$$

has the recurrence relation

$$r_{k+1}(\alpha) = (k\alpha + 1)r_k(\alpha) + \alpha(1-\alpha)r'_k(\alpha) \qquad (23)$$

and the first few values are as follows

$$\begin{aligned} r_1 &= 1 & r_3 &= 1 + 4\alpha + \alpha^2 \\ r_2 &= 1 + \alpha & r_4 &= 1 + 11\alpha + 11\alpha^2 + \alpha^3 \end{aligned}$$

$$r_5 = 1 + 26\alpha + 66\alpha^2 + 26\alpha^3 + \alpha^4$$

Notice that $r_k(0) = 1$, $r_k(1) = k!$, just as for the precise results.

6. EXPONENTIAL SUMS

The shape of the delay curves, from direct calculation, and also from Mellor's results, suggests representation in exponential sums. If

$$F(u) = A_1\, e^{-(1-\alpha)u/x_1} + A_2\, e^{-(1-\alpha)u/x_2} + \cdots \qquad (24)$$

then

$$M_k\,\frac{(1-\alpha)^k}{k!} = A_1x_1^k + A_2x_2^k + \cdots \qquad (25)$$

by a simple calculation. For k exponentials, $2k$ moments (including

M_0) may be fitted exactly by solution of $2k$ equations of form (25), as will be shown.

The first approximation ($k = 1$) is the order of arrival curve, say

$$F_1(u) = e^{-(1-\alpha)u}$$

which has $A_1 = x_1 = 1$, $A_k = x_k = 0$, $k > 1$, and matches M_0 and M_1.

The next approximation ($k = 2$) is determined from equations

$$\begin{aligned} A_1 \quad + A_2 \quad &= 1 \\ A_1 x_1 + A_2 x_2 &= 1 \\ A_1 x_1^2 + A_2 x_2^2 &= R_2 \\ A_1 x_1^3 + A_2 x_2^3 &= R_3 \end{aligned}$$

Eliminating A_2 from successive pairs,

$$\begin{aligned} A_1 (x_1 - x_2) &= 1 - x_2 \\ A_1 x_1 (x_1 - x_2) &= R_2 - x_2 \\ A_1 x_1^2 (x_1 - x_2) &= R_3 - R_2 x_2 \end{aligned}$$

Eliminating A_1 from these,

$$\begin{aligned} x_1 + x_2 \qquad - x_1 x_2 &= R_2 \\ (x_1 + x_2) R_2 - x_1 x_2 &= R_3 \end{aligned} \tag{26}$$

or, writing $a_1 = x_1 + x_2$, $a_2 = x_1 x_2$, so that $x^2 - a_1 x + a_2 = (x - x_1)(x - x_2)$

$$\begin{aligned} a_1 \quad - a_2 &= R_2 \\ a_1 R_2 - a_2 &= R_3 \end{aligned} \tag{26a}$$

From the first of the second set of equations, and from symmetry (or from $A_1 + A_2 = 1$)

$$\begin{aligned} A_1 &= \frac{1 - x_2}{x_1 - x_2} \\ A_2 &= \frac{1 - x_1}{x_2 - x_1} \end{aligned} \tag{27}$$

Taking R_2 and R_3 from Table II, it turns out that

$$\begin{aligned} x_1^{-1} &= 1 - \sqrt{\alpha/2} = 2A_1 \\ x_2^{-1} &= 1 + \sqrt{\alpha/2} = 2A_2 \end{aligned} \tag{28}$$

TABLE II — MOMENT RATIOS, CALLS SERVED AT RANDOM

$$R_k(\alpha) = M_k(1-\alpha)^k/k!$$

$$R_1 = 1$$

$$R_2 = \frac{2}{2-\alpha}$$

$$R_3 = \frac{2(2+\alpha)}{(2-\alpha)^2}$$

$$R_4 = \frac{4(6+5\alpha-4\alpha^2-\alpha^3)}{(2-\alpha)^3(3-2\alpha)}$$

$$R_5 = \frac{4(36+60\alpha-59\alpha^2-24\alpha^3+15\alpha^4+2\alpha^5)}{(2-\alpha)^4(3-2\alpha)^2}$$

$$R_6 = \frac{8f_6(\alpha)}{(2-\alpha)^5(3-2\alpha)^3(4-3\alpha)}$$

$$R_7 = \frac{8f_7(\alpha)}{(2-\alpha)^6(3-2\alpha)^4(4-3\alpha)^2}$$

$$f_6(\alpha) = 432 + 972\alpha - 2016\alpha^2 - 437\alpha^3 + 1790\alpha^4 - 528\alpha^5 - 196\alpha^6 + 67\alpha^7 + 6\alpha^8$$

$$f_7(\alpha) = 10368 + 34560\alpha - 89208\alpha^2 - 32772\alpha^3 + 177926\alpha^4 - 104287\alpha^5 - 29260\alpha^6 + 43876\alpha^7 - 9158\alpha^8 - 2039\alpha^9 + 588\alpha^{10} + 36\alpha^{11}$$

and the second approximation is

$$2F_2(u) = (1-\sqrt{\alpha/2})\, e^{-u(1-\alpha)(1-\sqrt{\alpha/2})} + (1+\sqrt{\alpha/2})\, e^{-u(1-\alpha)(1+\sqrt{\alpha/2})} \tag{29}$$

which turns out to be a good fit for α roughly less than 0.7. Curiously the corresponding Mellor approximation has a more complicated expression.

Following the same procedure for three exponentials, it turns out that the correspondent to the set of equations (26a) is

$$\begin{aligned} a_1R_2 - a_2 \quad + a_3 \quad &= R_3 \\ a_1R_3 - a_2R_2 + a_3 \quad &= R_4 \\ a_1R_4 - a_2R_3 + a_3R_2 &= R_5 \end{aligned} \tag{30}$$

with $a_1 = x_1 + x_2 + x_3$, $a_2 = x_1x_2 + x_1x_3 + x_2x_3$, $a_3 = x_1x_2x_3$, that is, the symmetric functions.

Using Table II for values of the R's, it is found that

$$
\begin{aligned}
a_1 &= (18 - 7\alpha - 2\alpha^2)(2 - \alpha)^{-1}(3 - 2\alpha)^{-1} \\
a_2 &= 18 \qquad (2 - \alpha)^{-1}(3 - 2\alpha)^{-1} \\
a_3 &= 6 \qquad (2 - \alpha)^{-1}(3 - 2\alpha)^{-1}
\end{aligned}
\tag{31}
$$

x_1, x_2 and x_3 are then the roots of the cubic equation

$$x^3 - a_1x^2 + a_2x - a_3 = 0$$

The coefficients A_i, $i = 1, 2, 3$ are determined from equations like

$$A_1 = \frac{R_2 - (x_2 + x_3) + x_2x_3}{(x_1 - x_2)(x_1 - x_3)} \tag{32}$$

For the fourth approximation, matching 8 moments, the equations for the symmetric functions are

$$
\begin{aligned}
a_1R_3 - a_2R_2 + a_3 \quad - a_4 \quad &= R_4 \\
a_1R_4 - a_2R_3 + a_3R_2 - a_4 \quad &= R_5 \\
a_1R_5 - a_2R_4 + a_3R_3 - a_4R_2 &= R_6 \\
a_1R_6 - a_2R_5 + a_3R_4 - a_4R_3 &= R_7
\end{aligned}
\tag{33}
$$

and x_1, x_2, x_3 and x_4 are roots of the quartic equation

$$x^4 - a_1x^3 + a_2x^2 - a_3x + a_4 = 0$$

Coefficients A_i are determined from equations like

$$A_1 = \frac{R_3 - (x_2 + x_3 + x_4)R_2 + (x_2x_3 + x_2x_4 + x_3x_4) - x_2x_3x_4}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} \tag{34}$$

It may be noted that

$$
\begin{aligned}
x_2 + x_3 + x_4 &= a_1 - x_1 \\
x_2x_3 + x_2x_4 + x_3x_4 &= a_2 - x_1(a_1 - x_1) \\
x_2x_3x_4 &= a_3 - x_1[a_2 - x_1(a - x_1)] = a_4x_1^{-1}
\end{aligned}
$$

which gives the general structure.

It is worth noting that equations (33) may be used to determine the R's if the a's may be determined otherwise. As a matter of fact, they have led to the determination of R_6 and R_7 in the following way. The

results for $k = 2$ and 3 suggest that

$$a_4 = 4!(2 - \alpha)^{-1}(3 - 2\alpha)^{-1}(4 - 3\alpha)^{-1}$$

$$a_3 = 4a_4$$

Then by the first two of equations (33)

$$a_1R_3 - a_2R_2 = R_4 - a_3 \quad + a_4$$

$$a_1R_4 - a_2R_3 = R_5 - a_3R_2 + a_4$$

the solutions of which are

$$a_1 = 4(24 - 23\alpha + 3\alpha^3) \qquad [(2 - \alpha)(3 - 2\alpha)(4 - 3\alpha)]^{-1}$$

$$a_2 = 2(72 - 23\alpha - 10\alpha^2 - 3\alpha^3)[(2 - \alpha)(3 - 2\alpha)(4 - 3\alpha)]^{-1}$$

By the last two of equations (33), R_6 and R_7 are determined to be the values given in Table II, which have been verified independently. Note that for $\alpha = 0$, both R_6 and R_7 are 1, and for $\alpha = 1$, $R_6 = 6!$, $R_7 = 7!$

Table III tabulates, for $k = 2$ to 5, for convenience in avoiding fractions the symmetric functions b_{kj} related to those above by

$$b_{kj} = D_ka_j$$

with, as before,

$$D_k = (2 - \alpha)(3 - 2\alpha) \cdots [k - (k - 1)\alpha]$$

and $a_0 = 1$. The functions for $k = 5$ were obtained by a process like

TABLE III — SYMMETRIC FUNCTIONS FOR EXPONENTIAL SUMS OF CALLS SERVED AT RANDOM

$k = 2$	$b_{20} = 2 - \alpha$	$k = 3$	$b_{30} = 6 - 7\alpha + 2\alpha^2$
	$b_{21} = 4$		$b_{31} = 18 - 7\alpha - 2\alpha^2$
	$b_{22} = 2$		$b_{32} = 18$
			$b_{33} = 6$
$k = 4$	$b_{40} = 24 - 46\alpha + 29\alpha^2 - 6\alpha^3$		
	$b_{41} = 96 - 92\alpha + 12\alpha^3$		
	$b_{42} = 144 - 46\alpha - 20\alpha^2 - 6\alpha^3$		
	$b_{43} = 96$		
	$b_{44} = 24$		
$k = 5$	$b_{50} = 120 - 326\alpha + 329\alpha^2 - 146\alpha^3 + 24\alpha^4$		
	$b_{51} = 600 - 978\alpha + 329\alpha^2 + 146\alpha^3 - 72\alpha^4$		
	$b_{52} = 1200 - 978\alpha - 172\alpha^2 + 78\alpha^3 + 72\alpha^4$		
	$b_{53} = 1200 - 326\alpha - 172\alpha^2 - 78\alpha^3 - 24\alpha^4$		
	$b_{54} = 600$		
	$b_{55} = 120$		

TABLE IV — SYMMETRIC FUNCTIONS FOR EXPONENTIAL SUMS, MELLOR APPROXIMATION

$k = 2$	$a_1 = 3+\alpha$ $a_2 = 2$	$k = 3$	$a_1 = 6 + 3\alpha$ $a_2 = 11 + 5\alpha + 2\alpha^2$ $a_3 = 6$
$k = 4$	$a_1 = 10 + 6\alpha$ $a_2 = 35 + 26\alpha + 11\alpha^2$ $a_3 = 50 + 26\alpha + 14\alpha^2 + 6\alpha^3$ $a_4 = 24$		
$k = 5$	$a_1 = 15 + 10\alpha$ $a_2 = 85 + 80\alpha + 35\alpha^2$ $a_3 = 225 + 200\alpha + 125\alpha^2 + 50\alpha^3$ $a_4 = 274 + 154\alpha + 94\alpha^2 + 54\alpha^3 + 24\alpha^4$ $a_5 = 120$		
$k = 6$	$a_1 = 21 + 15\alpha$ $a_2 = 175 + 190\alpha + 85\alpha^2$ $a_3 = 735 + 855\alpha + 585\alpha^2 + 225\alpha^3$ $a_4 = 1624 + 1604\alpha + 1194\alpha^2 + 704\alpha^3 + 274\alpha^4$ $a_5 = 1764 + 1044\alpha + 684\alpha^2 + 444\alpha^3 + 264\alpha^4 + 120\alpha^5$ $a_6 = 720$		

that sketched above, and without determining R_8 and R_9. Notice that

$$\begin{aligned} b_{kj} &= k!\binom{k}{j}, & \alpha &= 0 \\ &= j!\binom{k}{j}^2, & \alpha &= 1 \end{aligned}$$

which may be proved independently. All values in Table III satisfy the recurrence relation

$$\begin{aligned} b_{kj} = [k - (k - 1)\alpha]b_{k-1,j} + [k + (k - 1)\alpha]b_{k-1,j-1} \\ - (k - 1)^2\alpha b_{k-2,j-2} \end{aligned} \tag{35}$$

which also satisfies the boundary relations for $\alpha = 0$ and 1 given above for all values of k.

The corresponding symmetric functions for the Mellor approximation are given in Table IV. These have the recurrence relation

$$a_{kj} = a_{k-1,j} + [k + (k - 1)\alpha]a_{k-1,j-1} - (k - 1)^2\alpha a_{k-2,j-2} \tag{36}$$

For $\alpha = 0$, the values are the signless Stirling numbers of the first kind, that is, the numbers given by the expansion of

$$(1 + x)(1 + 2x) \cdots (1 + kx).$$

For $\alpha = 1$, the results are the same as for the exact case, as given above.

TABLE V — APPROXIMATIONS TO DELAY FUNCTION $F(u)$ FOR RANDOM SERVICE

α	v							
	1	2	4	6	8	10	12	14
Two Exponentials								
0.1	.3590	.1351	.0220	.0041	.0008	.0002		
0.2	.3490	.1344	.0256	.0058	.0015	.0004	.0001	
0.3	.3392	.1332	.0291	.0079	.0023	.0007	.0002	.0001
0.4	.3292	.1315	.0325	.0101	.0033	.0011	.0004	.0002
0.5	.3190	.1293	.0357	.0125	.0046	.0017	.0006	.0003
0.6	.3085	.1265	.0386	.0151	.0061	.0025	.0010	.0004
0.7	.2978	.1232	.0412	.0177	.0078	.0035	.0015	.0007
0.8	.2868	.1193	.0434	.0203	.0097	.0047	.0022	.0011
0.9	.2756	.1148	.0451	.0229	.0118	.0061	.0031	.0016
Three Exponentials								
0.1	.3586	.1354	.0219	.0040	.0008	.0002		
0.2	.3491	.1356	.0254	.0057	.0014	.0004	.0001	
0.3	.3393	.1358	.0288	.0074	.0022	.0007	.0002	.0001
0.4	.3291	.1360	.0322	.0092	.0030	.0011	.0004	.0002
0.5	.3186	.1363	.0358	.0112	.0040	.0016	.0007	.0003
0.6	.3071	.1359	.0392	.0133	.0050	.0022	.0010	.0005
0.7	.2951	.1354	.0428	.0156	.0063	.0028	.0014	.0007
0.8	.2822	.1344	.0466	.0181	.0077	.0036	.0018	.0010
0.9	.2683	.1325	.0504	.0210	.0094	.0045	.0023	.0013

7. NUMERICAL RESULTS

Table V gives both two-exponential and three-exponential 4 decimal approximations to the delay function $F(u)$ for

$$\alpha = 0.1(0.1)0.9(0.1 \text{ to } 0.9 \text{ in steps of } 0.1)$$

and for

$$u(1 - \alpha) = 1(1)2(2)14,$$

in the same abbreviated notation.* The variable $v = u(1 - \alpha)$ is introduced to reduce the spread of these tables. It will be noticed that, as expected, the two orders of approximation agree closely for small values of α; indeed, only for the three largest values of α are the differences appreciable from the engineering standpoint.

* The results for two exponentials, some of those for three-exponentials, and all special results given below, bave been obtained by Miss Marian Darville, whom I also tbank for her careful drawing of the curves. The entire three-exponential table has been computed independently by Miss Lennon.

For $\alpha = 0.9$, results for four exponentials have also been obtained and compare with those of Table V as follows (k = number of exponentials):

k	v							
	1	2	4	6	8	10	12	14
2	.2756	.1148	.0451	.0229	.0118	.0061	.0037	.0016
3	.2683	.1325	.0504	.0210	.0094	.0045	.0023	.0013
4	.2748	.1402	.0483	.0195	.0091	.0047	.0026	.0015

It is somewhat surprising that two exponentials should do as well as they do for large values of v (in fact for $v = 12$ and 14 better than three); a similar behavior appears in the following comparison of approximations on the Mellor basis, again for $\alpha = 0.9$

k	v							
	1	2	4	6	8	10	12	14
2	.2725	.1115	.0446	.0237	.0129	.0070	.0038	.0021
4	.2671	.1379	.0502	.0207	.0097	.0051	.0029	.0018
6	.2777	.1408	.0477	.0205	.0102	.0054	.0031	.0018

From these comparisons, it appears a relatively small number of exponentials is sufficient for engineering purposes. The curves of Fig. 1 are those for three exponentials, for uniformity.

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